

## ON GRAPHS WITH A METRIC END SPACE

KERSTIN WAAS

ABSTRACT. R. Diestel conjectured that an infinite graph contains a topologically end-faithful forest if and only if its end space is metrizable. We prove this conjecture for uniform end spaces.

### 1. INTRODUCTION

An important structural aspect of an infinite graph is the convergence pattern of its 1-way infinite paths. This is formalized by the concept of ends and the associated end space. Since the end structure of a forest is particularly simple, one is interested in classifying the graphs whose end space is reflected faithfully by a subtree or subforest.

In particular the end space of a forest is metrizable, so graphs with an end-faithful subforest have a metrizable end space. This led to the question of Diestel (1990) whether the converse also holds: *Does every graph with metrizable (topological) end space contain a (topologically) end-faithful forest?*

In this paper we prove the analogue of the above for uniform end spaces: Every graph with a metrizable (uniform) end space has a (uniformly) end-faithful subforest.

### 2. PRELIMINARIES

**2.1. Terminology.** In the following let  $G$  be a connected infinite graph. By  $V(G)$  we denote the set of vertices and by  $E(G)$  the set of edges of  $G$ . A *ray*  $T$  in  $G$  is a 1-way infinite path in  $G$ . A *tail* of a ray  $T$  is a connected infinite subgraph of  $T$ . Two or more paths are *independent* if their interiors are disjoint. For  $X, Y \subseteq G$  we call a path  $P \subset G$  an  $X$ – $Y$  *path* if its endvertices are in  $X$  and  $Y$ , respectively, and all inner vertices are in  $G \setminus (X \cup Y)$ . A subgraph  $X \subset G$  is *finitely* or *infinitely linked* to a subgraph  $Y \subset G$  in  $G$ , respectively, if there are finitely or infinitely many pairwise disjoint  $X$ – $Y$  paths in  $G$ , respectively.

A finite vertex set  $S \subset V(G)$  *separates two rays*  $T, T' \subseteq G$  in  $G$  if any two tails of  $T$  and  $T'$  belong to different components of  $G - S$ . We call two rays  $T, T' \subseteq G$  *end-equivalent* (or briefly *equivalent*), denoted by  $T \sim T'$ , if they are infinitely linked in  $G$ . In other words,  $T$  is equivalent to  $T'$  if there is no finite vertex set  $S \subset V(G)$  that separates  $T$  and  $T'$  in  $G$ . The relation  $\sim$  is an equivalence relation on the set

---

Received by the editors January 20, 1997.

1991 *Mathematics Subject Classification*. Primary 05C10.

*Key words and phrases*. Infinite graph, topological end space, uniformly end-faithful, topologically end-faithful.

Supported by *Deutsche Forschungsgemeinschaft*.

of rays of  $G$ . The equivalence classes of this relation are called *ends*; the set of all ends is called the *end space*  $\Omega(G)$  of  $G$ .

A subgraph  $H \subseteq G$  *contains* an end  $\omega \in \Omega(G)$  if  $H$  contains a tail of every ray in  $\omega$  and if in  $H$  every two rays of  $\omega$  are equivalent. If  $S \subseteq V(G)$  is a finite set of vertices, then every end of  $G$  is contained in a component of  $G - S$ . We call two ends  $\omega, \omega' \in \Omega(G)$  *equivalent with respect to  $S$*  if they are contained in the same component of  $G - S$  and denote this by  $(\omega \sim_S \omega')$ . Otherwise we say that  $S$  *separates these ends*.

Let  $\eta : \Omega(H) \rightarrow \Omega(G)$  be the mapping that maps every end of  $\Omega(H)$  to that end of  $\Omega(G)$  that is a superset of it. The subgraph  $H$  is called *end-faithful* if the mapping  $\eta$  is bijective. The sets  $V_S := \{(\omega, \omega') \in \Omega(G) \times \Omega(G) \mid \omega \sim_S \omega'\}$ , where  $S$  is taken over all finite subsets of  $V(G)$ , form a base of a uniform structure on  $\Omega(G)$ ; we denote this base by  $\mathcal{V}_G$ . This uniform structure induces a topology on  $\Omega(G)$ . We call the uniform space  $\Omega(G)$  *uniform end space* and the associated topological space  $\Omega(G)$  *topological end space*. Since for every two ends of  $G$  there exists a finite set of vertices that separates these two ends, we have  $\bigcap_{V_S \in \mathcal{V}_G} V_S = \{(\omega, \omega) \in \Omega(G) \times \Omega(G)\}$ , i.e. the uniform space  $\Omega(G)$  is separated. A subgraph  $H$  of  $G$  is called *topologically end-faithful* if the map  $\eta$  is a homeomorphism between the topological spaces  $\Omega(H)$  and  $\Omega(G)$ , i.e. if  $\eta$  is bijective and  $\eta$  as well as  $\eta^{-1}$  are continuous. The subgraph  $H$  is called *uniformly end-faithful* if  $\eta$  is an isomorphism between the uniform spaces  $\Omega(H)$  and  $\Omega(G)$ , i.e. if  $\eta$  is bijective and  $\eta$  as well as  $\eta^{-1}$  are uniformly continuous.

If  $B \subseteq G$  is a tree with root  $v$ , we define a partial order on the vertex set  $V(B)$  of  $B$ , the tree order  $<_B$ : For  $x, y \in V(B)$  we set  $x <_B y$  if  $x$  lies in the unique  $v$ - $y$  path in  $B$ . The tree  $B$  is called *normal* in  $G$  if the initial vertex and the end-vertex of any  $B$ - $B$  path in  $G$  are comparable in the tree order.

We denote a countably infinite complete graph by  $K_{\aleph_0}$  and call its vertices *branch vertices*. Any subdivided  $K_{\aleph_0}$  is denoted by  $TK_{\aleph_0}$ . A fat  $TK_{\aleph_0}$  is a subdivided  $K_{\aleph_0}$ , whose branch vertices are pairwise linked by  $\aleph_1$  independent paths. (Two paths are called *independent* if they have no common inner vertices.) The vertices of infinite degree are called *branch vertices*. Let  $A \subseteq G$  be a  $K_{\aleph_0}$  or a fat  $TK_{\aleph_0}$ , respectively; we say  $A$  is *contained* in a subgraph  $H \subseteq G$  if  $A \cap H$  is a  $K_{\aleph_0}$  or a fat  $TK_{\aleph_0}$ , respectively. Since all rays of a  $K_{\aleph_0}$  or fat  $TK_{\aleph_0}$ , respectively, are equivalent in  $G$ , we say a  $K_{\aleph_0}$  or fat  $TK_{\aleph_0}$ , respectively, *belongs to an end*.

A finite vertex set  $S \subseteq V(G)$  *separates a vertex  $x \in V(G)$  and a ray  $T \subseteq G$  in  $G$*  if every  $x$ - $T$  path in  $G$  meets  $S$ . The vertex set  $S$  *separates a vertex  $x \in V(G)$  and a  $K_{\aleph_0}$  or fat  $TK_{\aleph_0}$  in  $G$ , respectively*, if the vertex  $x$  does not lie in that component of  $G - S$  that contains the  $K_{\aleph_0}$  or fat  $TK_{\aleph_0}$ . Moreover  $S$  *separates an end  $\omega \in \Omega(G)$  and a vertex  $v \in V(G)$  in  $G$*  if  $\omega$  is contained in a component of  $G - S$  that does not contain the vertex  $x$ . The vertex set  $S$  *separates an end  $\omega$  and a ray  $T \subseteq V(G)$  in  $G$*  if a tail of  $T$  and the end  $\omega$  are contained in different components of  $G - S$ . We will use the following notation: Let  $H \subseteq G$  be a subgraph; then we denote by  $N_G(H)$  the set of neighbours of  $H$  in  $G \setminus H$ . By  $H_G$  we denote the subgraph of  $G$  that consists of  $H \cup N_G(H)$  and all  $H$ - $N_G(H)$  edges of  $G$ .

**2.2. The uniform end structure.** In a tree, clearly, two rays are equivalent if and only if they have a tail in common. Thus an end-faithful tree or forest represents the end-structure of a graph in a very simple way. So it is of interest to study the following question of Halin, [4]: *Does every connected graph contain an end-faithful spanning tree, or at least an end-faithful forest?* Counterexamples of Seymour &

Thomas [11] and of Thomassen [12] show that this is not the case. However, Halin [4] proved that every countable connected graph contains an end-faithful spanning tree.

The end space of a tree  $B$  with root  $v$  is, as mentioned before, metrizable; consider for example the following metric: Given distinct ends  $\omega, \omega'$ , let  $x_{\omega, \omega'}$  be the vertex in  $B$  that separates  $\omega$  and  $\omega'$  and that has maximal distance from  $v$ , say  $n(\omega, \omega')$ ; then  $d(\omega, \omega') := 1/(n(x_{\omega, \omega'}) + 1)$ . It is easy to show that this metric generates the topological end space. This leads to the following conjecture stated by Diestel, [2], as a problem: *A graph contains a topologically endfaithful forest if and only if its end space is metrizable.* An answer to this problem is given by an observation of Diestel [2]: Every normal spanning tree of a connected graph is topologically end-faithful. But which graphs contain a normal spanning tree? Jung proved [8] that this is true for every countable connected graph. Furthermore every connected graph that does not contain  $TK_{\aleph_0}$  and every connected graph that contains neither a  $K_{\aleph_0}$  nor a fat  $TK_{\aleph_0}$  contain a normal spanning tree, as shown by Halin [5]. Hence these graphs contain a topologically end-faithful tree.

For a metrizable uniform space the associated topological space is also metrizable. Since every uniformly end-faithful subgraph is also topologically end-faithful, also the following version of Diestel's question is of interest: *Does every graph with a metrizable uniform end space contain a uniformly end-faithful forest?* This is the case:

**Theorem 1.** *Every connected graph with metrizable uniform end space contains a uniformly end-faithful tree.*

It should be noted that, unlike in the topological case, the converse of this theorem does not hold. It easy to construct a forest whose uniform end space is not metrizable [13].

### 3. CONSTRUCTION OF A $K_{\aleph_0}$ - AND FAT $TK_{\aleph_0}$ -FREE SUBGRAPH

**3.1. Construction of  $S^*$ .** For the proof of Theorem 1 we need the following two theorems:

**Theorem 2** ([5]). *Every connected graph that does not contain a  $K_{\aleph_0}$  or fat  $TK_{\aleph_0}$  contains a normal spanning tree.*

**Theorem 3** ([2]). *Every normal spanning tree is uniformly end-faithful.*

We now sketch the proof of Theorem 1: Throughout the paper let  $G$  be a fixed connected graph with a metrizable uniform end-space  $\Omega(G)$ . Our aim is to construct a connected uniformly end-faithful subgraph of  $G$  that contains neither  $K_{\aleph_0}$  nor fat  $TK_{\aleph_0}$ . Then, by Theorems 2 and 3 this subgraph, and hence also  $G$ , contain a uniformly end-faithful tree. To do this we first construct a countable vertex set  $S^* \subset V(G)$  such that  $G - S^*$  decomposes into components that contain a  $K_{\aleph_0}$  or fat  $TK_{\aleph_0}$  only if they are one-ended. In each of these components we select a ray. We then link all the other components of  $G - S^*$  containing ends with the selected rays via  $S^*$ . In this way we obtain a connected subgraph of  $G$  that is uniformly end-faithful to  $G$  and does not contain a  $K_{\aleph_0}$  or fat  $TK_{\aleph_0}$ .

The above construction is carried out in two steps: First we construct a graph  $G' \subseteq G$  that does not contain a  $K_{\aleph_0}$ ; we then construct a graph  $G'' \subseteq G'$  that also does not contain a fat  $TK_{\aleph_0}$ . Then we show that  $G''$  is uniformly end-faithful to  $G$ .

Let us recall a few topological properties that we need for the construction of  $S^*$ . By assumption the uniform end space is metrizable. So, by the following theorem, it has a countable base for the uniformity:

**Theorem 4** ([10]). *A uniform space is metrizable if and only if it is separated and has a countable base for the uniformity.*

If a filter has a countable base, then, as one easily verifies, every base contains a countable base. For this reason the neighbourhood base  $\mathcal{V}_G$  introduced in Section 1 contains a countable base  $\mathcal{V}'_G$ . We set  $\tilde{S} := \bigcup_{V_S \in \mathcal{V}'_G} S$  and  $\tilde{\mathcal{V}}_G := \{V_S | S \subset \tilde{S}, |S| < \aleph_0\}$ . Since  $\tilde{S}$  is countable,  $\tilde{\mathcal{V}}_G$  is also countable. Furthermore  $\tilde{\mathcal{V}}_G$  is a filter base of  $\Omega(G)$ , since it is a superset of the filter base  $\mathcal{V}'_G$  of  $\Omega(G)$ . Thus we have proved the following lemma:

**Lemma 1.** *There exists a countable vertex set  $\tilde{S} \subset V(G)$  such that the filter base  $\tilde{\mathcal{V}}_G := \{V_S | S \subset \tilde{S}, |S| < \aleph_0\}$  generates the same uniform structure on  $\Omega(G)$  as the filter base  $\mathcal{V}_G = \{V_S | S \subset V(G), |S| < \aleph_0\}$ .*

*Remark 1.* If we say that  $\tilde{S}$  generates the uniform structure, we mean this in the sense of Lemma 1. This means in particular that for every two ends of  $G$  there exists a finite subset of  $\tilde{S}$  that separates these ends. Furthermore we have the following: If an end  $\omega \in \Omega(G)$  is separated in  $G$  from all other ends by a finite vertex set  $S \subset V(G)$ , then there exists a finite subset  $S' \subseteq \tilde{S}$  that separates  $\omega$  in  $G$  from all other ends. To prove this remember that only one element of the set  $V_S \in \mathcal{V}_G$  contains  $\omega$ , namely  $(\omega, \omega)$ . Since  $\tilde{\mathcal{V}}_G$  is a filter base, there exists a set  $V_{S'} \in \tilde{\mathcal{V}}_G$  with  $V_{S'} \subseteq V_S$ . In this set the end  $\omega$  occurs also only in  $(\omega, \omega)$ . So the finite vertex set  $S' \subseteq \tilde{S}$  separates the end  $\omega$  from all other ends in  $G$ .

If  $\tilde{S}$  is finite, every end of  $G$  is contained in an one-ended component of  $G - \tilde{S}$ . Hence, in this case we can select from every end of  $G$  a ray that begins in  $\tilde{S}$  such that all these rays are pairwise disjoint (except for their initial vertices). We link the vertices in  $\tilde{S}$  by a finite tree  $B \subset G$ . The subgraph of  $G$  consisting of  $B$  and the chosen rays then contains neither a  $K_{\aleph_0}$  nor a fat  $TK_{\aleph_0}$ , since all vertices of infinite degree lie in  $B$ ; but  $V(B)$  is finite. Since  $V(B)$  separates pairwise all ends of this graph, it is clearly uniformly end-faithful to  $G$ . For this reason we assume in the following that  $\tilde{S}$  is infinite.

**Construction of  $S^*$ :** We now construct inductively the vertex set  $S^*$ : Let  $\tilde{S} := \{b_1, b_2, \dots\}$  and  $S_1 := \{b_1\}$ . The graph  $G - S_1$  decomposes into components. Since  $S_1$  is finite, every  $K_{\aleph_0}$  or fat  $TK_{\aleph_0}$  is contained in one of these components. Let  $\mathcal{C}_1$  be the set of components that contain a  $K_{\aleph_0}$  or fat  $TK_{\aleph_0}$ . By  $X_1$  we denote the set of all vertices in  $\tilde{S}$  that lie in one-ended components of  $\mathcal{C}_1$ . Consider now the set of all vertices in  $\tilde{S}$  that belong to a component of  $\mathcal{C}_1$  or that are adjacent to such a component. From this set we delete all vertices of  $X_1$  and denote the resulting set by  $S^1$ . To formalize this we set  $S^1 := \bigcup_{C \in \mathcal{C}_1} (V(C) \cup N_G(C)) \cap (\tilde{S} \setminus X_1)$ .

Suppose  $X_1, \dots, X_{k-1}$  and  $S^1, \dots, S^{k-1}$  have been defined for some  $k > 0$ . Let  $S_k := \{b_1, \dots, b_k\}$ . Then  $G - (S_k \setminus \bigcup_{i < k} X_i)$  decomposes into components. Since  $S_k \setminus \bigcup_{i < k} X_i$  is finite, every  $K_{\aleph_0}$  or fat  $TK_{\aleph_0}$  is contained in one of these components. Let  $\mathcal{C}_k$  be the set of those components that contain a  $K_{\aleph_0}$  or fat  $TK_{\aleph_0}$ . By  $X_k$  we denote the set of all vertices in  $\tilde{S}$  that lie in one-ended components of  $\mathcal{C}_k$ . Then let  $S^k := \bigcup_{C \in \mathcal{C}_k} (V(C) \cup N_G(C)) \cap (\tilde{S} \setminus \bigcup_{i \leq k} X_i)$ . Finally we set  $S^* := \bigcap_{i \in \mathbb{N}} S^i$ .

*Remark 2.* By the construction of  $S^*$  we have  $S_i \setminus \bigcup_{k < i} X_k \subseteq S_j \setminus \bigcup_{k < j} X_k$  for all  $i < j$ , since  $(S_l \setminus \bigcup_{k < l} X_k) \cap X_l = \emptyset$  and  $S_{l+1} \supseteq S_l$  for all  $l \in \mathbb{N}$ . This roughly means that the graph decomposes step by step into finer components. Thus we have  $S^i \supseteq S^j$  for  $i < j$ .

In the construction of  $G'$  the graph  $G - S^*$  decomposes into components, some of which contain ends. But not every end of  $G$  must be contained in one of these components, since  $S^*$  may be infinite. So we distinguish different types of ends: On one hand, those ends of  $G$  whose rays are infinitely linked to  $S^*$  in  $G$  and on the other hand, those ends whose rays are only finitely linked to  $S^*$ . By definition of an ‘end’ it is clear that an end belongs either to the first or to the second type. Furthermore the ends of the second type are contained in a component of  $G - S^*$ .

Our aim is to construct a subgraph of  $G$  that contains neither a  $K_{\aleph_0}$  nor a fat  $TK_{\aleph_0}$ . So we distinguish such ends to which a  $K_{\aleph_0}$  or fat  $TK_{\aleph_0}$  belongs and such ends for which this is not the case.

In the construction of  $G'$  we take those components of  $G - S^*$ , that contain ends and the vertices of  $S^*$  and link them by edges. By an arbitrary selection of these edges it is quite possible that we generate a  $K_{\aleph_0}$  or fat  $TK_{\aleph_0}$  that is not contained in one of the components of  $G - S^*$ . So we have to take care which edges we select in the construction of  $G'$ . Therefore we distinguish those ends that are contained in a component of  $G - S^*$  with finitely many neighbours in  $S^*$  and those ends that are contained in a component with infinitely many neighbours in  $S^*$ . We now characterize the ends of  $G$  by four types:

**Type 1:** An end  $\omega \in \Omega(G)$  is of Type 1 if in  $G$  every ray of  $\omega$  is infinitely linked to  $S^*$ .

**Type 2:** An end  $\omega \in \Omega(G)$  is of Type 2 if in  $G$  every ray of  $\omega$  is only finitely linked to  $S^*$  and a  $K_{\aleph_0}$  or fat  $TK_{\aleph_0}$  belongs to  $\omega$ .

**Type 3:** An end  $\omega \in \Omega(G)$  is of Type 3 if in  $G$  every ray of  $\omega$  is only finitely linked to  $S^*$ , neither a  $K_{\aleph_0}$  nor a fat  $TK_{\aleph_0}$  belongs to  $\omega$  and the set of neighbours  $N_G(C)$  of the component  $C$  of  $G - S^*$  that contains  $\omega$  is finite.

**Type 4:** An end  $\omega \in \Omega(G)$  is of Type 4, if in  $G$  every ray of  $\omega$  is only finitely linked to  $S^*$ , neither a  $K_{\aleph_0}$  nor a fat  $TK_{\aleph_0}$  belongs to  $\omega$  and the set of neighbours  $N_G(C)$  of the component  $C$  of  $G - S^*$  that contains  $\omega$  is infinite.

*Remark 3.* Clearly each end of  $G$  belongs to one of these four types of ends.

We show now that every end of Type 2, 3 and 4 is contained in a component of  $G - S^*$  and furthermore we show that such components are one-ended in the case of Type 2.

**Lemma 2.** *Let  $\omega \in \Omega(G)$  be an end such that in  $G$  every ray of  $\omega$  is only finitely linked to  $S^*$ . Then  $\omega$  is contained in a component of  $G - S^*$ . If furthermore a  $K_{\aleph_0}$  or fat  $TK_{\aleph_0}$  belongs to  $\omega$ , this component is one-ended and has only finitely many neighbours in  $S^*$ .*

*Proof.* Let  $\omega \in \Omega(G)$  be an end such that in  $G$  every ray of  $\omega$  is only finitely linked to  $S^*$ . Then every ray  $T \in \omega$  contains a tail in  $G - S^*$ , since  $T$  contains at most finitely many vertices of  $S^*$ . Furthermore in  $G - S^*$  each two rays  $T, T' \in \omega$  are equivalent, since from any infinite set of pairwise disjoint  $T-T'$  paths at most finitely many of these paths contain vertices of  $S^*$ . This means that  $\omega$  is contained in a component of  $G - S^*$ .

Now let  $\omega \in \Omega(G)$  be such that in  $G$  every ray of  $\omega$  is only finitely linked to  $S^*$  and such that a  $K_{\aleph_0}$  or fat  $TK_{\aleph_0}$  belongs to  $\omega$ . Then there exists a finite vertex set that separates  $\omega$  in  $G$  from all other ends. Suppose this were not the case and let  $T$  be an arbitrary ray of  $\omega$ . Since there are only finitely many pairwise disjoint  $T$ - $S^*$  paths in  $G$ , there exists a finite vertex set  $S'$  that separates  $\omega$  and  $S^*$  in  $G$  (that means  $\omega$  is contained in a component of  $G - S'$  that does not contain a vertex of  $S^*$ ). Furthermore there exists an end  $\omega' \in \Omega(G)$  with  $\omega' \neq \omega$  that is contained in the same component  $C_\omega$  of  $G - S'$  as  $\omega$ . Then  $V(C_\omega) \cap S^* = \emptyset$ , but  $V(C_\omega) \cap \tilde{S} \neq \emptyset$ , since  $\tilde{S}$  generates the uniform structure (see Remark 1).

Since  $S'$  is finite,  $C_\omega$  contains branch vertices of the  $K_{\aleph_0}$  and fat  $TK_{\aleph_0}$  that belong to  $\omega$ . We denote by  $H$  the set of these branch vertices. Let  $W$  be an arbitrary  $H$ - $\tilde{S}$  path in  $C$ . By  $v_W$  we denote the initial vertex of  $W$  in  $H$  and by  $e_W$  the end-vertex of  $W$  in  $\tilde{S}$ . The vertices  $e_W$  and  $v_W$  are not separable by a finite subset of  $\tilde{S} \setminus \{e_W\}$  and hence the vertex  $e_W$  is not separable from the associated  $K_{\aleph_0}$  or fat  $TK_{\aleph_0}$ .

Now we show that  $e_W \notin \bigcup_{i \in \mathbb{N}} X_i$ . Suppose the converse and let  $j \in \mathbb{N}$  be minimal with  $e_W \in X_j$ . Then by definition of  $X_j$  the vertex  $e_W$  lies in an one-ended component  $C^j$  of  $G - (S_j \setminus \bigcup_{i < j} X_i)$  that contains a  $K_{\aleph_0}$  or fat  $TK_{\aleph_0}$ . Let  $\omega^*$  be the end of  $G$  that is contained in  $C^j$ . Then the  $K_{\aleph_0}$  or fat  $TK_{\aleph_0}$  belongs to  $\omega^*$ . We will show that  $C^j$  is a component of  $G - S^*$ . Therefore we consider again the construction of  $S^*$ : For every  $k > j$  let  $C^k$  be the component of  $G - (S_k \setminus \bigcup_{i < k} X_i)$  that contains  $\omega^*$ . As shown in Remark 2 we have  $S_k \setminus \bigcup_{i < k} X_i \supseteq S_j \setminus \bigcup_{i < j} X_i$  and hence  $C^k \subseteq C^j$  for all  $k > j$ . On the other hand we have  $V(C^j) \cap \tilde{S} \subseteq X_j$  by definition of  $X_j$ . So  $C^k \supseteq C^j$  for all  $k > j$ , since  $C^k$  is a component of  $G - (S_k \setminus \bigcup_{i < k} X_i)$ . Hence  $C^k = C^j$  for  $k > j$ . That means  $C^j$  is a component of  $G - (S_k \setminus \bigcup_{i < k} X_i)$  for all  $k \geq j$ . Since  $C^j$  contains a  $K_{\aleph_0}$  or fat  $TK_{\aleph_0}$  we have  $N_G(C^j) \subseteq S^k$  for all  $k \geq j$ . Furthermore we have  $S^k \supseteq S^j$  for all  $k < j$  (see Remark 2). So we have  $N_G(C^j) \subseteq \bigcap_{i \in \mathbb{N}} S^i = S^*$ , which means  $C^j$  is a component of  $G - S^*$ .

By assumption  $C^j$  is one-ended. So the vertex  $e_W$  lies in the one-ended component  $C^j$  of  $G - S^*$ . But as we assumed above  $e_W$  lies in the component  $C_\omega$  of  $G - S'$  that contains more than one end but no vertices of  $S^*$ . This means  $C_\omega$  is a subgraph of the one-ended component  $C^j$ . Because of this contradiction we have  $e_W \notin \bigcup_{i \in \mathbb{N}} X_i$ .

Since there is no finite subset of  $\tilde{S} \setminus \{e_W\}$  that separates  $e_W$  in  $G$  from the branch vertex  $v_W$ , we have  $e_W \in S^k$  for all  $k \in \mathbb{N}$ . Thus,  $e_W \in S^*$ , in contradiction to  $S^* \cap V(C_\omega) = \emptyset$ . Hence there exists a finite vertex set of  $G$  that separates  $\omega$  in  $G$  from all the other ends.

Since  $\tilde{S}$  generates the uniform structure on  $\Omega(G)$ , there exists a finite subset of  $\tilde{S}$  that separates  $\omega$  in  $G$  from all the other ends (see Remark 1). Thus, there exists a minimal  $j > 0$  such that  $\omega$  is contained in an one-ended component  $K$  of  $G - S_j$ . We show now that  $K$  is also a component of  $G - (S_j \setminus \bigcup_{i < j} X_i)$ . This is clear for  $j = 1$ . Hence in the following let  $j > 1$ . For all  $k \leq j$  let  $K^k$  be the component of  $G - (S_k \setminus \bigcup_{i < k} X_i)$  that contains  $\omega$ . Furthermore let  $K^{j-1}$  be a component of  $G - S_{j-1}$ . Since  $S_j \setminus \bigcup_{i < j} X_i \supseteq S_{j-1} \setminus \bigcup_{i < j-1} X_i$ , we have  $K^j \supseteq K^{j-1}$ . So  $K^j$  is also a component of  $G - (S_j \setminus X_{j-1})$ . But since  $j-1 < j$ , the component  $K^{j-1}$  contains more than one end. Hence  $V(K^{j-1}) \cap X_{j-1} = \emptyset$ . Thus  $K^j$  is also a component of

$G - S_j$ , so  $K^j = K$ . That means  $K$  is the component of  $G - (S_j \setminus \bigcup_{i < j} X_i)$  that contains  $\omega$ . We have shown above that a one-ended component of  $G - (S_j \setminus \bigcup_{i < j} X_i)$  that contains a  $K_{\aleph_0}$  or fat  $TK_{\aleph_0}$  is also a component of  $G - S^*$ . Thus  $K$  is a component of  $G - S^*$  and  $N_G(K)$  is finite.  $\square$

*Remark 4.* As we have just seen, the ends of Type 3 and Type 4 are contained in components of  $G - S^*$ . We denote the set of these components by  $\mathcal{C}$ . Furthermore the ends of Type 2 are contained in one-ended components of  $G - S^*$  that have a finite neighbourhood in  $S^*$ . We denote the set of these components by  $\mathcal{C}^*$ . By the definition of  $\mathcal{C}$  and  $\mathcal{C}^*$  we have  $\mathcal{C} \cap \mathcal{C}^* = \emptyset$ . Furthermore we have  $\bigcup_{i \in \mathbb{N}} X_i \subseteq \bigcup_{C \in \mathcal{C}^*} C$ , by the proof of Lemma 2.

**3.2. Construction of  $G'$  by removing every  $K_{\aleph_0}$  of  $G$ .** We consider the subgraph  $\bigcup_{C \in \mathcal{C}} C$  and link the components of this graph by the vertices of  $S^* := \{s_0, s_1, s_2, \dots\}$ : Set  $t_0 := s_0$  and  $t_1 := s_1$  and choose an arbitrary but fixed  $t_1$ - $t_0$  path  $W_1$  in  $G$ ; furthermore let  $r(x) := 1$  for all  $x \in V(W_1)$ . Suppose the vertices  $t_0, \dots, t_{k-1}$  and the paths  $W_1, \dots, W_{k-1}$  have been chosen for some  $k > 1$  and for all  $i < k$  let  $r(x) = i$  for all  $x \in V(B_i \setminus B_{i-1})$ ; here  $B_i := \bigcup_{j \leq i} W_j$  and  $B_{i-1} := \bigcup_{j \leq i-1} W_j$ . Let  $t_k$  be the vertex with minimal index in  $S^* \setminus B_{k-1}$ . We choose a  $t_k$ - $B_{k-1}$  path  $W_k$  by the following rules:

- R1) The end vertex  $v_k \in B_{k-1}$  of  $W_k$  should have maximal possible value  $r(v_k)$ .
- R2) When selecting  $W_k$  according to rule R1) select  $v_k$  such that in  $B_{k-1}$  it has maximal distance to the vertex  $t_0$ .

Then  $B := \bigcup_{i \in \mathbb{N}} W_i$  is a tree with root  $v := t_0$ . We now consider the ends of Type 2: As we have seen in Lemma 2 each of these ends is contained in a one-ended component of  $G - S^*$ . As mentioned before we denote the set of these components by  $\mathcal{C}^*$ . For every end  $\omega$  of Type 2 we now select a ray  $T_\omega$  that lies in the component  $C_\omega \in \mathcal{C}^*$  that contains  $\omega$ . Let  $t_n$  be the vertex of  $N_G(C_\omega)$  with maximal index (by Lemma 2 the set  $N_G(C_\omega)$  is finite). We link the ray  $T_\omega$  and the tree  $B$  by a  $T_\omega$ - $t_n$  path  $W_\omega$  in  $C_\omega^G$  (here  $C_\omega^G$  denotes the graph that consists of  $C_\omega \cup N_G(C_\omega)$  and all  $C_\omega$ - $N_G(C_\omega)$  edges of  $G$ ). Now we add all  $C$ - $S^*$  edges of the components  $C \in \mathcal{C}$ . We denote the resulting graph by  $G'$ .

*Remark 5.* The graph  $G'$  is composed of four different types of subgraphs of  $G$ : First it contains the end-containing components of  $G - S^*$  that contain neither a  $K_{\aleph_0}$  nor a  $TK_{\aleph_0}$ ; second it contains subgraphs of the components of  $G - S^*$  that contain a  $K_{\aleph_0}$  or  $TK_{\aleph_0}$ ; third it contains edges between  $S^*$  and all these components; and fourth it contains a tree  $B$  linking the vertices of  $S^*$ . By construction this graph is connected. However, one should mention that the tree  $B$  is not necessarily disjoint to the components of  $G - S^*$ .

**Definition 1** ([1]). Let  $H$  be a tree with root  $b$  and let  $x \in V(H)$ . Then the unique  $x$ - $b$  path in  $H$  is called *down-closure*  $[x]$  of  $x$ .

**Definition 2.** Let  $H$  be a tree and  $x, y \in V(H)$ . Then the vertex  $x$  lies *above* the vertex  $y$  if  $y$  lies in the down-closure of  $x$ . The vertex  $x$  lies *below*  $y$  if  $x$  lies in the down-closure of  $y$ .

We now prove several lemmata, that we need later on:

**Lemma 3.** Let  $H$  be a rooted tree and  $W$  a path in  $H$  and  $v$  a vertex in  $H \setminus W$ . If  $v$  lies in the down-closure of a vertex of  $W$ , then  $v$  lies in the down-closure of every vertex of  $W$ .

*Proof.* Let  $t$  and  $t'$  be two vertices of  $W$ ; then the unique  $t$ - $t'$  path  $W_{t,t'}$  in  $H$  is a subpath of  $W$ . This path consists of the vertices of  $[t] \setminus [t']$ ,  $[t'] \setminus [t]$  and the vertex of  $[t] \cap [t']$  with maximal distance to the root. We now assume that there exists a vertex  $v$  from  $H \setminus W$  and two vertices  $t$  and  $s$  of  $W$  such that  $v$  lies in the down-closure of the vertex  $t$  but not in the down-closure of the vertex  $s$ . Then  $v$  lies in  $[t] \setminus [s]$  and hence in the path  $W_{t,s}$ . Thus  $v$  lies also in  $W$ , in contradiction to the assumption.  $\square$

**Lemma 4.** *The tree  $B$  is normal in  $G$ .*

*Proof.* We use the same notation as in the construction of  $G'$ . For all  $i \in \mathbb{N}$  we denote by  $t_i$  the initial vertex and by  $v_i$  the end-vertex of the path  $W_i$ . Suppose there exists a  $B$ - $B$  path  $W$  in  $G$  with end-vertices  $h, h' \in V(B)$  such that neither  $h$  nor  $h'$  lies in the down-closure of the other. Let  $r(h) = i$ ,  $r(h') = k$  and w.l.o.g.  $i > k$ . Furthermore let  $m > k$  be minimal, such that the unique  $t_m$ - $h$  path of  $B$  contains only vertices  $x$  with  $r(x) > k$ . We denote this path by  $\tilde{W}$ . Due to the minimality of  $m$  we have  $r(v_m) \leq k$ . Thus  $v_m$  does not lie in  $\tilde{W}$ . But, since  $v_m$  is in the down-closure of  $t_m$ , by Lemma 3 the vertex  $v_m$  is also in the down-closure of  $h$ . If  $r(v_m) < k = r(h')$ , then  $W_m$  does not obey rule R1), since the  $t_m$ - $h'$  path  $W' := \tilde{W} \cup W$  contains only vertices  $x$  with  $r(x) \geq k$  and such vertices that do not lie in  $B$ . Hence  $r(v_m) = k$ . Then  $v_m$  is in the down-closure of  $h'$  or conversely. The latter means that  $h'$  lies in the down-closure of  $h$ , since  $v_m$  lies in the down-closure of  $h$ . This is a contradiction to our assumption.

Thus we may assume that  $v_m$  lies in the down-closure of  $h'$ . But then the path  $W_m$  does not obey rule R2), since the path  $W' := \tilde{W} \cup W$  contains (except for the end vertex  $h'$ ) only vertices  $x$  with  $r(x) > k$  and vertices that do not lie in  $B$  and ends in  $h'$ , where  $h'$  has larger distance to the root as  $v_m$ . But this is a contradiction to the construction of  $B$ .  $\square$

Since  $B$  is normal in  $G$ , there are two interesting structural properties that we need for the proof of Theorem 1:

**Lemma 5** ([1]). *For any two vertices  $x, y \in V(B)$  we have  $[x] \cap [y]$  separates  $x$  and  $y$  in  $G$ .*

**Lemma 6** ([1]). *If  $W = x_1 \dots x_n$  is a  $B$ - $B$  path in  $G$ , then  $x_1$  and  $x_n$  are comparable in the tree order  $<_B$ .*

Lemma 5 and Lemma 6 are equivalent to results of the paper *Normal Tree Orders For Infinite Graphs* by J.-M. Brochet and R. Diestel [1]. The proofs given there may be easily modified to prove the above presented versions.

**Lemma 7.** *Let  $H \supseteq B$  be an arbitray subgraph of  $G$  and let  $T \subseteq H$  be a ray that is infinitely linked to  $S^*$  in  $H$ . Then in  $H$  the ray  $T$  is equivalent to a ray of  $B$ .*

*Proof.* Since  $T$  is infinitely linked to  $S^*$  in  $H$ , we can choose infinitely many pairwise disjoint  $T$ - $S^*$  paths  $W_i$  in  $H$ . For all  $i \in \mathbb{N}$  we denote by  $v_i$  the initial vertex of the path  $W_i$  in  $T$  and by  $e_i$  the end-vertex in  $S^*$ . Furthermore we set  $E := \{e_1, e_2, \dots\}$  and denote by  $v$  the root of  $B$  (see construction of  $G'$ ). For all  $i \in \mathbb{N}$  we denote by  $\tilde{W}_i$  the unique  $e_i$ - $v$  path in  $B$  and set  $B' := \bigcup_{i \in \mathbb{N}} \tilde{W}_i$ . Then  $B'$  is a tree with root  $v$ . Since the paths  $W_i$  are pairwise disjoint, the set  $E$  is infinite; hence also  $B'$  is infinite. Then by König's Theorem [9]  $B'$  contains either a vertex of infinite degree or a ray. Suppose the first is the case; then there exist infinitely many pairwise



disjoint (except for the vertex  $v$ )  $v$ - $E$  paths in  $B'$  whose associated end-vertices  $e_{i_j}, j \in \mathbb{N}$ , lie in  $B'$  above  $v$ . W.l.o.g. let these be labeled in such a way that the associated end vertices  $v_{i_j}$  of the paths  $W_{i_j}$  are ordered on  $T$  in the order of their labels. For each two vertices  $v_{i_{2j}}, v_{i_{2j+1}}$  let  $W_{i_{2j}, i_{2j+1}}$  be the unique  $v_{i_{2j}}-v_{i_{2j+1}}$  path in  $T$ . Due to the ordering of the  $v_{i_j}$  on  $T$  these paths are pairwise disjoint. Since the paths  $W_i$  are pairwise disjoint for all  $i \in \mathbb{N}$  and contain no vertices of  $T$  (except for the end-vertices), also the  $e_{i_{2j}}-e_{i_{2j+1}}$  paths  $\bar{W}_{i_{2j}} := W_{i_{2j}} \cup W_{i_{2j}, i_{2j+1}} \cup W_{i_{2j+1}}$  are pairwise disjoint.

Since  $B$  is normal in  $G$ , also  $B' \subseteq B$  is normal in  $G$ . Then, by Lemma 6 every path  $\bar{W}_{i_{2j}}$  must contain a vertex of  $[e_{i_{2j}}] \cap [e_{i_{2j+1}}] = [v]$ . Since  $[v]$  is finite (it even contains only the vertex  $v$ ), this is a contradiction to the pairwise disjointness of the paths  $\bar{W}_{i_{2j}}$ .

So  $B'$  contains a ray  $R$ ; w.l.o.g we assume that it starts in the root of  $B'$ . Let us show that there are infinitely many pairwise disjoint  $T$ - $R$  paths in  $H$ . Suppose every set of pairwise disjoint  $T$ - $R$  paths in  $H$  is finite. Let  $\mathcal{W}$  be a set of such paths; then  $V(\mathcal{W})$  is finite, say  $|V(\mathcal{W})| = n$ . Then there exists a vertex  $x \in V(R) \setminus V(\mathcal{W})$  such that no vertex of  $V(\mathcal{W})$  lies above  $x$ . Since  $R$  is a ray, all except finitely many vertices of  $R$  lie in  $B$  above  $x$ . We show now that infinitely many vertices  $e_{i_j} \in E$ ,  $j \in \mathbb{N}$ , lie in  $B$  above  $x$ . Suppose not; then either there exists a vertex  $y \in V(R) \setminus E$  that lies in  $B$  above  $x$  but no vertex in  $E$  lies above  $y$  or no vertex of  $E$  lies above  $x$  (in this case we set  $y := x$ ). But this is a contradiction to the construction of  $B'$ , since  $y$  lies in at least one of the paths  $\bar{W}_i$ . Hence infinitely many vertices of  $E$  lie in  $B'$  above  $x$ . We select  $n+1$  of these vertices  $e_{i_1}, \dots, e_{i_{n+1}}$ . From assumption the associated  $T$ - $e_{i_j}$  paths  $W_{i_j}$  are pairwise disjoint. Since  $V(\mathcal{W})$  contains only  $n$  distinct vertices, at least one of these paths, say  $W_{i_k}$ , is disjoint to the paths of  $\mathcal{W}$ . Since all vertices of  $R$  that lie above  $x$  (including  $x$ ) do not lie in  $V(\mathcal{W})$ , the unique  $e_{i_k}-x$  path  $W$  in  $B$  does not contain a vertex in  $V(\mathcal{W})$ . Thus also the path  $W_{i_k} \cup W$  does not contain a vertex of  $V(\mathcal{W})$ . This path, however, contains a  $T$ - $R$  path as subpath (since it starts in  $T$  and ends in  $R$ ) which is disjoint to  $V(\mathcal{W})$ . But this is a contradiction to the maximality of  $\mathcal{W}$ . Thus there exist infinitely many pairwise disjoint  $T$ - $R$  paths in  $H$ , and hence,  $R$  is in  $H$  equivalent to  $T$ .  $\square$

*Remark 6.* By Lemma 7 the tree  $B$  contains a ray of every end of Type 1 and hence also  $G' \supseteq B$  does so. Hence by Lemma 2 and Lemma 7 each end of  $G$  is represented by a ray in  $G'$ . Later on we will show that in  $G'$  an end of  $G$  does not split into several ends.

We show now that  $G'$  does not contain a  $K_{\aleph_0}$ :

**Lemma 8.** *The components of  $\mathcal{C}$  and  $\mathcal{C}^* \cap G'$  do not contain a branch vertex of a  $K_{\aleph_0}$  or fat  $TK_{\aleph_0}$  of  $G'$ .*

*Proof.* By construction of  $G'$ , we have  $G' \setminus B \subseteq \mathcal{C} \cup (\mathcal{C}^* \cap G')$ . Furthermore for  $C_\omega \in \mathcal{C}^*$  the subgraph  $C_\omega \cap G'$  consists of a ray  $T_\omega$ , a  $T_\omega$ - $S^*$  path (except for its end-vertex in  $S^*$ ) and (again except for their end-vertices in  $S^*$ ) of at most finitely many  $S^*-S^*$  paths of  $B$  (since  $N_G(C_\omega)$  is finite and  $B$  is composed of  $S^*-S^*$  paths). Since all vertices of  $C_\omega \cap G'$  have only finite degree, no branch vertex of a  $K_{\aleph_0}$  or fat  $TK_{\aleph_0}$  of  $G'$  lies in  $C_\omega \cap G'$ .

Let  $C$  be a component of  $\mathcal{C}$ . Suppose a branch vertex of a  $K_{\aleph_0}$  or  $TK_{\aleph_0}$  of  $G'$  lies in  $C$ . We denote by  $\omega$  the end to which the  $K_{\aleph_0}$  or fat  $TK_{\aleph_0}$  belongs. By the definition of  $\mathcal{C}$ , the component  $C$  contains an end  $\omega'$  to which neither a  $K_{\aleph_0}$  nor

a fat  $TK_{\aleph_0}$  belongs, so  $\omega \neq \omega'$ . Since  $\tilde{S}$  generates the uniform structure of  $\Omega(G)$ , there exists a finite subset of  $\tilde{S}$  that separates  $\omega$  and  $\omega'$  in  $G$  (see Remark 2). Since  $C$  contains  $\omega'$  and a branch vertex of a  $K_{\aleph_0}$  or fat  $TK_{\aleph_0}$  belonging to  $\omega$ , we have  $\tilde{S} \cap V(C) \neq \emptyset$ .

We denote by  $H$  the set of all branch vertices of all  $K_{\aleph_0}$  or fat  $TK_{\aleph_0}$  that lie in  $C$ . Then  $C$  contains an  $H$ - $\tilde{S}$  path and we denote by  $v$  its end-vertex in  $\tilde{S}$ . Since  $C \in \mathcal{C}$  and  $\bigcup_{i \in \mathbb{N}} X_i \subseteq \bigcup_{C \in \mathcal{C}^*} C$  and  $\mathcal{C} \cap \mathcal{C}^* = \emptyset$ , we have  $v \notin \bigcup_{k \in \mathbb{N}} X_k$ . Then in every step of the construction of  $S^*$  the vertex  $v$  lies either in the same component as a  $K_{\aleph_0}$  or fat  $TK_{\aleph_0}$  or is adjacent to that component. Thus  $v \in S^k$  for all  $k \in \mathbb{N}$  and hence  $v \in S^*$ . But this is a contradiction to  $v \in V(C)$ .  $\square$

**Lemma 9.** *The graph  $G'$  does not contain a  $K_{\aleph_0}$ .*

*Proof.* By Lemma 8 the branch vertices of a  $K_{\aleph_0}$  of  $G'$  lie in  $G' \setminus (\mathcal{C} \cup \mathcal{C}^*) = B \setminus (\mathcal{C} \cup \mathcal{C}^*)$ . Since this subgraph is a forest, it does not contain a  $K_{\aleph_0}$ .  $\square$

**3.3. Construction of  $G''$ .** By Lemma 9 the graph  $G'$  does not contain a  $K_{\aleph_0}$ . But it is quite possible that  $G'$  contains some fat  $TK_{\aleph_0}$ . We will show that their branch vertices lie in  $S^*$ . The vertices of  $S^*$  lie in  $B$  and in  $G'$  they are adjacent to the components of  $\mathcal{C} \cup \mathcal{C}^*$ . But each two branch vertices of a fat  $TK_{\aleph_0}$  are linked by uncountably many pairwise independent paths. Since the tree  $B$  is countable, in  $G'$  uncountably many pairwise independent paths can only run through the components of  $\mathcal{C} \cup \mathcal{C}^*$ . Thus, if we delete in  $G'$  sufficiently many edges between the components of  $\mathcal{C} \cup \mathcal{C}^*$  and the vertices in  $S^*$  such that there are only countably many pairwise disjoint  $S^*-S^*$  paths in the resulting graph  $G''$ , then this graph neither contains a  $K_{\aleph_0}$  nor a fat  $TK_{\aleph_0}$ .

For the construction of  $G''$  we need some further lemmata:

**Lemma 10.** *The branch vertices of a fat  $TK_{\aleph_0}$  of  $G'$  lie in  $S^*$ .*

*Proof.* By Lemma 8 the branch vertices of a fat  $TK_{\aleph_0}$  of  $G'$  lie in  $G' \setminus (\mathcal{C} \cup \mathcal{C}^*) = B \setminus (\mathcal{C} \cup \mathcal{C}^*)$ . By construction the tree  $B$  is countable. But a branch vertex of a fat  $TK_{\aleph_0}$  has uncountable degree. Thus only such vertices of  $B \setminus (\mathcal{C} \cup \mathcal{C}^*)$  that are adjacent in  $G'$  to the components of  $\mathcal{C} \cup \mathcal{C}^*$  can be branch vertices of a fat  $TK_{\aleph_0}$ . But only the vertices in  $S^*$  have this property.  $\square$

**Lemma 11.** *Each component  $C \in \mathcal{C}$  with infinite  $N_G(C)$  contains vertices of  $\tilde{S}$ .*

*Proof.* Let  $C$  be a component of  $\mathcal{C}$  with infinite  $N_G(C)$ . If  $C$  contains rays from several ends, we clearly have  $V(C) \cap \tilde{S} \neq \emptyset$ , since  $\tilde{S}$  generates the uniform structure (see Remark 1). Otherwise by definition of  $\mathcal{C}$  the component  $C$  contains exactly one end  $\omega$  of Type 4. Each ray  $T \in \omega$  has by definition only finitely many pairwise disjoint  $T$ - $S^*$  paths in  $G$ . Thus, there exists a finite vertex set of  $C$  that separates  $\omega$  from  $S^*$ . Suppose this were not the case and let  $T \in \omega$  be chosen arbitrary but fixed. Let  $W$  be a maximal set of pairwise disjoint  $T$ - $S^*$  paths in  $G$ . Since  $V(W)$  is finite,  $\omega$  is contained in a component of  $G - V(W)$ . This component does not contain vertices of  $S^*$ , since otherwise there exists a  $T$ - $S^*$  path that is disjoint to the paths of  $W$ , which is a contradiction to the maximality of  $W$ .

Hence there exists a finite vertex set  $S \subseteq V(G)$  that separates in  $G$  the end  $\omega$  and the vertex set  $S^*$ . Since  $N_G(C) \subseteq S^*$ , the component of  $G - S$  that contains  $\omega$  is a subgraph of the component  $C$ . So  $S$  separates the end  $\omega$  in  $G$  from all other ends, since  $C$  is one-ended. Then there exists a finite subset of  $\tilde{S}$  that separates  $\omega$  in  $G$

from all other ends, since  $\tilde{S}$  generates the uniform structure (see Remark 1). For all  $j \in \mathbb{N}$  let  $S_j$  be defined as in the construction of  $S^*$  and let  $i$  be minimal such that  $S_i$  separates  $\omega$  in  $G$  from all other ends. Suppose this is not the case for  $S_i \setminus \bigcup_{j < i} X_j$ . Let  $\omega'$  be an end that is not separated from  $\omega$  and let  $T \in \omega$  and  $T' \in \omega'$  be rays in  $G - (S_i \setminus \bigcup_{j < i} X_j)$ . Then there exists a  $T$ - $T'$  path  $W$  in  $G - (S_i \setminus \bigcup_{j < i} X_j)$ . Since  $S_i$  separates the two ends,  $W$  contains a vertex  $x \in \bigcup_{j < i} X_j$ . By definition of the  $X_j$ , there is a minimal  $k < i$  such that the vertex  $x$  lies in a component of  $G - (S_k \setminus \bigcup_{j < k} X_j)$  that contains exactly one end and a  $K_{\aleph_0}$  or fat  $TK_{\aleph_0}$  belong to this end. Since this is not the case for  $\omega$  ( $\omega$  is of Type 4), the end  $\omega$  is not contained in this component. Then  $T \cup W$  must contain a vertex of  $S_k \setminus \bigcup_{j < k} X_j$ . But  $S_k \setminus \bigcup_{j < k} X_j \subseteq S_i \setminus \bigcup_{j < i} X_j$  for  $k < i$  (see Remark 2), i.e.  $T \cup W$  contains a vertex of  $S_i \setminus \bigcup_{j < i} X_j$ , in contradiction to the choice of  $W$  and  $T$ . Hence the vertex set  $S_i \setminus \bigcup_{j < i} X_j$  separates  $\omega$  in  $G$  from all other ends.

Suppose now that  $C$  does not contain a vertex of  $\tilde{S}$ . Then  $C$  does not contain a vertex of  $S_i \setminus \bigcup_{j < i} X_j \subseteq \tilde{S}$ . But since this set is finite, it separates at most finitely many vertices of  $N_G(C)$  from  $C$ , and thus from  $\omega$ . The other vertices are separated from all the other ends. Since neither a  $K_{\aleph_0}$  nor a fat  $TK_{\aleph_0}$  belongs to  $\omega$  ( $\omega$  is of Type 4),  $C$  contains neither a  $K_{\aleph_0}$  nor fat  $TK_{\aleph_0}$ . Then all except finitely many vertices of  $N_G(C)$  lie in a component of  $G - (S_i \setminus \bigcup_{j \leq i} X_j)$  that does not contain a  $K_{\aleph_0}$  or fat  $TK_{\aleph_0}$ . So in the construction of  $S^*$  these vertices of  $N_G(C)$  lie not in  $S^i$  and hence also not in  $S^*$ , which is a contradiction to the definition of  $N_G(C)$ .  $\square$

*Remark 7.* Since the vertex set  $\tilde{S}$  is countable and the components of  $\mathcal{C}$  are pairwise disjoint, by Lemma 11 there are only countably many components  $C \in \mathcal{C}$  with infinite  $N_G(C)$ .

**Definition 3** ([3]). Let  $X$  be a subgraph of a graph  $G$  and let  $v$  be a vertex of  $G \setminus X$ , such that  $G$  contains infinitely many pairwise disjoint (except for  $v$ )  $v$ - $X$  paths. Then the subgraph of  $G$  consisting of these paths is called a  $v$ - $X$  fan in  $G$ .

**Lemma 12** ([3]). Let  $U$  and  $C$  be disjoint subgraphs of a graph  $G$  such that  $U$  is infinite, every vertex of  $U$  has a neighbour in  $C$  and  $C$  is connected. Then  $G$  either contains a  $v$ - $U$  fan for a  $v$  of  $C$ , or there exists a ray  $R \subset C$  with infinitely many pairwise disjoint  $R$ - $U$  paths in  $G$ .

In the following we denote by  $C^G$  for every component  $C \in \mathcal{C}$  the subgraph of  $G$  that consists of  $C \cup N_G(C)$  and all  $C$ - $N_G(C)$  edges of  $G$ .

**Lemma 13.** For all  $C \in \mathcal{C}$  the graph  $C^G$  does not contain a  $v$ - $S^*$  fan with  $v \in V(C)$ .

*Proof.* Suppose there exists a component  $C \in \mathcal{C}$  such that  $C^G$  contains a  $v$ - $S^*$  fan with  $v \in V(C)$ . Then  $N_G(C)$  must be infinite and  $C$  contains by Lemma 11 vertices of  $\tilde{S}$ . Thus we can choose a  $v$ - $\tilde{S}$  path in  $C$ . Let  $s$  be the end-vertex of this path in  $\tilde{S}$ . Then no finite subset of  $\tilde{S} \setminus \{s\}$  separates  $s$  and  $N_G(C)$ . Since in any step of the construction of  $S^*$  only a finite set of vertices is deleted in  $G$ , either the vertex  $s$  lies in each of these steps in a component that contains infinitely many vertices of  $N_G(C) \subseteq S^*$  or  $s$  is adjacent to such a component. Since all but finitely many vertices of  $S^*$  lie in any step of the construction of  $S^*$  in a component that contains a  $K_{\aleph_0}$  or fat  $TK_{\aleph_0}$ , in any of these steps either the vertex  $s$  lies also in such a component or it is adjacent to such a component. But then  $s \in S^*$ , in contradiction to  $s \in V(C)$ .  $\square$

**Corollary 1.** *For  $C \in \mathcal{C}$  the neighbourhood  $N_G(C)$  is infinite if and only if  $C$  contains a ray  $T$  with infinitely many pairwise disjoint  $T-S^*$  paths in  $G$ .*

*Proof.* Let  $C \in \mathcal{C}$  be a component with infinite  $N_G(C)$ . By definition  $N_G(C)$  is disjoint to  $C$  and every vertex of  $N_G(C)$  has a neighbour in  $C$ . Furthermore  $C$  is connected. So by Lemma 12 and Lemma 13 there exists a ray  $T \subseteq C$  with infinitely many pairwise disjoint  $T-N_G(C)$  paths in  $G$ . Since  $N_G(C) \subseteq S^*$ , this proves the ‘if’ part of the proposition. To see the converse remember that  $V(C) \cap S^* = \emptyset$  but  $N_G(C) \subseteq S^*$ .  $\square$

**Definition 4.** Let  $\mathcal{R}$  be an infinite set of pairwise disjoint rays that belong to ends of Type 1 and that all start in a vertex  $v \in V(G)$ ; furthermore let all rays of  $\mathcal{R}$  be pairwise disjoint (except for the vertex  $v$ ). Then the subgraph of  $G$  that consists of all rays of  $\mathcal{R}$  is called a big star with center  $v$ .

**Lemma 14.** *The components  $C \in \mathcal{C}$  do not contain a big star.*

*Proof.* Suppose there exists a component  $C \in \mathcal{C}$  that contains a big star with center  $v$ . Since every ray  $T$  of a big star belongs to an end of Type 1, it has infinitely many pairwise disjoint  $T-S^*$  paths in  $G$ . Then via an inductive process indexed over the branches of the big star we can select infinitely many  $v-S^*$  paths which are pairwise disjoint except for the initial vertex  $v$ . But then  $C^G$  contains a  $v-S^*$  fan with  $v \in V(C)$ , in contradiction to Lemma 13.  $\square$

Now let  $C \in \mathcal{C}$  be an arbitrary but fixed component with infinite  $N_G(C)$ . Then by Corollary 1  $C$  contains a ray of an end of Type 1. Consider  $C$  as a subgraph of  $G'$  and let  $\Omega(C)$  be the end space of  $C$ . This end space consists of two sorts of ends: on one hand ends of Type 4 that are contained in  $C$  and on the other hand ends that are in  $G$  subsets of an end of Type 1. In the following we denote by  $\mathcal{E}_C$  the set of ends of the second sort.

**Definition 5.** Let  $E$  be a subset of  $\Omega(G)$ , and let  $M$  be a set of rays in  $G$  such that the map  $f : M \rightarrow E$  that maps every ray of  $M$  to that end of  $\Omega(G)$  that contains this ray is bijective. Then  $M$  is called a set of representatives of  $E$ .

*Remark 8.* Definition 5 means that  $M$  contains exactly one ray of every end of  $E$  but no further rays.

**Definition 6.** Let  $M$  be a set of rays in  $G$ . Then we call a countable vertex set  $Y \subset V(G)$  that contains infinitely many vertices of every ray in  $M$  an  $M$ -cover.

We now select one ray from every end of  $\mathcal{E}_C$ . Let  $\Gamma_C$  be the set of these rays. Then  $\Gamma_C$  is a set of representatives of  $\mathcal{E}_C$ . We show now that there exists a  $\Gamma_C$ -cover:

**Lemma 15.** *For each component  $C \in \mathcal{C}$  with infinite  $N_G(C)$  there exists a  $\Gamma_C$ -cover.*

*Proof.* Let  $C$  be a component of  $\mathcal{C}$  with infinite  $N_G(C)$  and let  $v_0 \in V(C)$  be an arbitrary but fixed vertex. For every ray  $T \in \Gamma_C$  we select a ray  $T_{v_0} \subset C$  that begins in  $v_0$  and that contains a tail of  $T$ . Then let  $\Gamma'_C := \{T_{v_0} | T \in \Gamma_C\}$ . Let “ $\subseteq$ ” be an order of  $\Gamma'_C$  and let  $\alpha$  be the ordinal number of  $(\Gamma'_C, \subseteq)$ . We set  $T := T_\beta$ , if  $T$  is in the position  $\beta$  of the order “ $\subseteq$ ” of  $\Gamma'_C$ .

Now for all  $\beta < \alpha$  we inductively define a tree  $B_\beta$ : Let  $B_0 := T_0$  and suppose that for arbitrary  $\beta > 0$  the trees  $B_\gamma$  are defined for all  $\gamma < \beta$ . Then we set  $B_\beta := B'_\beta$

if  $|V(B'_\beta \cap T_\beta)| = \infty$  and  $B_\beta := B'_\beta \cup xT_\beta$ , otherwise. Here  $B'_\beta := \bigcup_{\gamma < \beta} B_\gamma$  and  $x$  is the last vertex of  $T_\beta$  in  $B'_\beta$  and  $xT_\beta$  is the tail of  $T_\beta$  beginning in  $x$ . (The vertex  $x$  exists, since  $T_\beta$  begins in  $v_0 \in B_0 \subset B'_\beta$ .)

We set  $B := \bigcup_{\beta < \alpha} B_\beta$ . Then  $B$  is a tree and every  $T_\beta \in \Gamma'_C$  has infinitely many vertices in  $B$ . Since  $B$  is a subgraph of  $C$  by Lemma 14, every vertex in  $B$  has finite degree, i.e.  $V(B)$  is countable. But then the vertex set  $V(B)$  is a  $\Gamma'_C$ -cover, and hence also a  $\Gamma_C$ -cover.  $\square$

We now construct a subgraph  $G''$  of  $G'$  such that  $G''$  does not contain a fat  $TK_{\aleph_0}$ . As mentioned before this can be achieved by deleting sufficiently many edges between  $S^*$  and the components of  $\mathcal{C} \cup \mathcal{C}^*$ . For every component  $C \in \mathcal{C}$  we denote (in analogy to  $C^G$ ) by  $C^{G'}$  the subgraph of  $G'$  consisting of  $N_{G'}(C) \cup C$  and all  $C-N_{G'}(C)$  edges of  $G'$ . Since  $N_G(C) = N_{G'}(C)$  and since in the construction of  $G'$  all  $C-N_G(C)$  edges are inserted, we have  $C^G = C^{G'}$ .

**Construction of  $G''$ :** We delete in  $G'$  all edges between  $S^*$  and the components  $C \in \mathcal{C}$ , except those that lie in  $B$ . We consider first the components  $C \in \mathcal{C}$  with infinite  $N_G(C)$ : Let  $\mathcal{E}_C$  and  $\Gamma_C$  be defined as above. By Lemma 15 there exists a set  $Y_{\Gamma_C} \subseteq V(C)$  that is a  $\Gamma_C$ -cover. Let  $\mathcal{P} := Y_{\Gamma_C} \times N_G(C)$ . Since  $Y_{\Gamma_C}$  and  $N_G(C)$  are countable,  $\mathcal{P}$  is also countable. For every finite subset  $P$  of  $\mathcal{P}$  for which this is possible we now choose for every pair  $(a, b) \in P$  an  $a$ - $b$  path in  $C^{G'}$  such that these paths are pairwise independent. Two paths that are chosen to different finite subsets of  $\mathcal{P}$ , however, must not be independent. We now reinsert all  $N_G(C)$ - $C$  edges of all chosen paths. For the components  $C \in \mathcal{C}$  with finite  $N_G(C)$  we proceed as follows: Let  $t_{n(C)}$  be the vertex with maximal index in  $N_G(C)$ . For each of these components we insert a  $t_{n(C)}$ - $C$  edge. The resulting graph is the graph  $G''$ .

*Remark 9.* Since for every component  $C \in \mathcal{C}$  with infinite  $N_G(C)$  for every singleton of  $\mathcal{P}$  a path is chosen, every vertex of  $N_G(C)$  is connected with  $C$ . Furthermore for every component  $C \in \mathcal{C}$  with finite  $N_G(C)$  a  $C$ - $S^*$  edges is chosen. Hence  $G''$  is connected.

We now prove that the resulting graph  $G''$  contains neither a  $K_{\aleph_0}$  nor a fat  $TK_{\aleph_0}$ . The proof requires the following lemma:

**Lemma 16.** *Every set of pairwise independent  $S^*$ - $S^*$  paths of  $G''$  is countable.*

*Proof.* In  $G''$  the components  $C \in \mathcal{C}$  with finite  $N_G(C)$  are linked with  $N_G(C)$  by exactly one  $S^*$ - $C$  edge (apart from the edges of  $B$ ). Since  $B$  is countable and the components of  $\mathcal{C}$  are pairwise disjoint, at most countably many pairwise independent  $S^*$ - $S^*$  paths run through these components. By the same argument, this holds also for the components of  $\mathcal{C}^*$ . By Lemma 11 there are only countably many components  $C \in \mathcal{C}$  with infinite  $N_G(C)$ . In  $G''$  each of these components is linked with its neighbourhood only by countably many edges that lie not in  $B$ . Since  $B$  is countable, there exist only countably many pairwise independent  $S^*$ - $S^*$  paths in  $G''$  that run through these components. Altogether it follows that there are at most countably many pairwise independent  $S^*$ - $S^*$  paths in  $G''$ .  $\square$

**Corollary 2.** *The graph  $G''$  does not contain a  $K_{\aleph_0}$  or fat  $TK_{\aleph_0}$ .*

*Proof.* By Lemma 9 the graph  $G''$  does not contain a  $K_{\aleph_0}$ , since it is a subgraph of  $G'$ . In Lemma 10 we have shown that the branch vertices of every fat  $TK_{\aleph_0}$  of

$G'$  lie in  $S^*$ . Since for every pair  $(v, v')$  of branch vertices of a fat  $TK_{\aleph_0}$  there exist uncountably many pairwise independent  $v-v'$  paths, by Lemma 16 the graph  $G''$  does not contain a fat  $TK_{\aleph_0}$ .  $\square$

**3.4. Proof of the uniform end-faithfulness of  $G''$  and  $G$ .** In this chapter we prove that the graph  $G''$  is uniformly end-faithful to  $G$ . To do this recall the meaning of the term *uniformly end-faithful*. The subgraph  $G'' \subseteq G$  is called uniformly end-faithful if the map  $\eta : \Omega(G'') \rightarrow \Omega(G)$  that maps every end of  $G''$  to the end of  $G$  that it contains as subset is an isomorphism between the uniform spaces  $\Omega(G)$  and  $\Omega(G'')$ . We thus have to show that  $\eta$  is a bijection and that  $\eta$  as well as its inverse  $\eta^{-1}$  is uniformly continuous.

The map  $\eta$  is 1-1 if and only if two rays of  $G''$  that are equivalent in  $G$  are also equivalent in  $G''$ . The map  $\eta$  is onto if and only if  $G''$  contains a ray of every end of  $\Omega(G)$ .

The map  $\eta$  is uniformly continuous if and only if for every neighbourhood  $V$  of the uniform end-space  $\Omega(G)$  there exists a neighbourhood  $V''$  of the uniform end-space of  $\Omega(G'')$  such that  $V'' \subseteq \eta^{-1} \times \eta^{-1}(V)$ . Due to the filter-properties of the uniformity it suffices to show this for the elements of the base  $\mathcal{V}_G$ . But an element of the base  $\mathcal{V}_G$  has the form  $V_S = \{(\omega, \omega') \in \Omega(G) \times \Omega(G) \mid \omega \sim_S \omega'\}$ . The same holds for the corresponding filter base of the uniformity of  $\Omega(G'')$ . So it suffices to show that for every finite vertex set  $S \subset V(G)$  there exists a finite vertex set  $S' \subset V(G'')$  that separates those ends in  $G''$  whose associated ends are separated by  $S$  in  $G$ . Since  $G''$  is a subgraph of  $G$ , the vertex set  $S' := S \cap V(G'')$  has this property. Hence the map  $\eta$  is uniformly continuous for every subgraph of  $G$ . For the inverse map  $\eta^{-1}$  the situation is different; it is not always uniformly continuous. To summarize, we have to show the following:

- 1) The subgraph  $G''$  contains a ray from every end of  $\Omega(G)$  (that means  $\eta$  is onto);
- 2) Two rays of  $G''$  that are equivalent in  $G$  are also equivalent in  $G''$  (that means  $\eta$  is 1-1);
- 3) For every finite vertex set  $S \subset V(G'')$  there exists a finite vertex set  $S' \subset V(G)$ , such that  $S'$  separates those ends in  $G$  whose associated ends are separated by  $S$  in  $G''$  (that means  $\eta^{-1}$  is uniformly continuous).

The following two lemmata are needed to show the bijectivity of  $\eta$ . We denote by  $C^{G''}$  (in analogy to  $C^G$ ) the subgraph of  $G''$  that consists of  $C \cup N_{G''}(C)$  and all  $C-N_{G''}(C)$  edges of  $G''$ .

**Lemma 17.** *Let  $C$  be a component of  $\mathcal{C}$  and  $R$  a ray of  $C$  such that there are infinitely many pairwise disjoint  $R-S^*$  paths in  $C^G$ . Let  $\mathcal{W}$  be a set of such paths and let  $A$  be the set of their end-vertices in  $S^*$ . Then there exist infinitely many pairwise disjoint  $R-A$  paths in  $C^{G''}$ .*

*Proof.* By assumption  $N_G(C)$  is infinite and we define  $\mathcal{E}_C, \Gamma_C$  and  $Y_{\Gamma_C}$  as above. Since there exist infinitely many pairwise disjoint  $R-S^*$  paths in  $G$ , the ray  $R$  belongs to an end of Type 1. Then by definition of  $\mathcal{E}_C$  the ray  $R$  belongs to an end of  $\mathcal{E}_C$  and by definition of  $\Gamma_C$  there exists a ray  $T_R \in \Gamma_C$  that is equivalent to  $R$  in  $C$ . Hence there exist infinitely many pairwise disjoint  $T_R-A$  paths in  $C^G$ . Since  $Y_{T_R} := Y_{\Gamma_C} \cap V(T_R)$  is infinite by the definition of  $Y_{\Gamma_C}$ , there exist also infinitely many pairwise disjoint  $Y_{T_R}-A$  paths in  $C^G$ .

For every finite subset  $P \subset \mathcal{P} = Y_{\Gamma_C} \times N_G(C)$  for which this is possible in  $G$ , the graph  $G''$  contains a set of pairwise independent paths, whose initial and end-vertices can be identified with the pairs of  $P$ . Since  $A$  is a subset of  $N_G(C)$  and  $Y_{T_R}$  is a subset of  $Y_{\Gamma_C}$ , for every  $i \in \mathbb{N}$  there exists a set of pairwise disjoint  $Y_{T_R}-A$  paths in  $C^{G''}$  with cardinality strictly larger than  $i$ .

Suppose that in  $C^{G''}$  every set of pairwise disjoint  $T_R-A$  paths is finite. Then also every set of pairwise disjoint  $Y_{T_R}-A$  paths is finite. So we can select a finite set  $M$  of pairwise disjoint  $Y_{T_R}-A$  paths, such that there exists no set of such paths in  $C^{G''}$  that is a strict superset of  $M$ . Then  $V(M) := \bigcup_{w \in M} V(w)$  is finite, say  $|V(M)| = n$ , and it separates the vertex sets  $Y_{T_R}$  and  $A$  in  $C^{G''}$ . Thus there exist at most  $n$  pairwise disjoint  $Y_{T_R}-A$  paths in  $C^{G''}$ . But this contradicts the construction of  $G''$ . Hence there exist infinitely many pairwise disjoint  $T_R-A$  paths in  $C^{G''} \subseteq G''$  and, since  $R$  is equivalent to  $T_R$ , there exist also infinitely many pairwise disjoint  $R-A$  paths in  $C^{G''} \subseteq G''$ .  $\square$

**Lemma 18.** *Every two rays  $T, T' \subseteq B$  that are equivalent in  $G$  have a common tail.*

*Proof.* W.l.o.g let  $T$  and  $T'$  be rays starting at the root of  $B$ . Since they are equivalent in  $G$  there exists an infinite set  $W$  of pairwise disjoint  $T-T'$  paths  $W_i \subseteq G$ ,  $i \in \mathbb{N}$ . We denote by  $t_i$  the initial vertex of  $W_i$  in  $T$  and by  $t'_i$  the end-vertex of  $W_i$  in  $T'$ . Furthermore let  $v$  be the vertex in  $B$  with smallest distance to the root that separates  $T$  and  $T'$ . Then in  $B - \{v\}$  all tails of  $T$  lie in a component  $C$  whose vertices all lie above  $v$ . The same holds for  $T'$  and a component  $C' \neq C$  of  $B - \{v\}$ . Otherwise a vertex  $v'$  that lies below  $v$  would separate the two rays in  $B$ , in contradiction to the minimality of the distance between  $v$  and the root of  $B$ . Thus two vertices  $x \in V(C)$  and  $y \in V(C')$  are incomparable in the order  $<_B$ . Furthermore  $[x] \cap [y] = [v]$  for all  $x \in V(C)$ ,  $y \in V(C')$ .

Since  $[v]$  is finite, all but finitely many of the vertices  $t_i$  and  $t'_i$  lie in the components  $C$  and  $C'$ . But only finitely many of the paths  $W_i$  contain vertices of  $[v]$ , since these paths are pairwise disjoint. Thus there exists a path  $W_j = x_1 \dots x_n$ , whose end-vertices are incomparable with the tree order  $<_B$  and that contains no vertex of  $[t_j] \cap [t'_j] = [v]$ . Let  $x_k$  be the vertex of  $W_j \cap C$  with maximal index. Then  $k < n$ , since  $y = x_n \notin V(C)$ . Let  $x_l$  be the vertex with minimal index  $l > k$  of  $W_j \cap B$ . Since  $x_l$  does not lie in  $C$  and also not in  $[v]$ , the vertices  $x_l$  and  $x_k$  are incomparable in the tree order. Then  $x_k \dots x_l$  is a  $B-B$  path of  $G$  with (in the order  $<_B$ ) incomparable end-vertices. But this is a contradiction to Lemma 6.  $\square$

*First part of the proof of Theorem 1, bijectivity of  $\eta$ .* To prove that the map  $\eta : \Omega(G'') \rightarrow \Omega(G)$  is onto, we have to show that  $G''$  contains a ray of every end of  $\Omega(G)$ . We first consider the ends of Type 3 and 4: By construction the graph  $G''$  contains the components of  $\mathcal{C}$ . Since every end of Type 3 and 4 is contained in one of these components,  $G''$  contains rays from each of these ends. By Lemma 7 the tree  $B$  contains a ray from each end of Type 1. Since  $B$  is a subgraph of  $G''$ , it follows that  $G''$  contains also rays from each of these ends. By definition every end  $\omega$  of Type 2 is contained in a component  $C_\omega \in \mathcal{C}^*$ . By the construction of  $G'$  and  $G''$  we have  $V(C_\omega) \cap G'' \supseteq T_\omega$ , where  $T_\omega$  is a ray of  $\omega$ . Thus  $G''$  also contains rays from every end of Type 2. Since every end of  $G$  is of one of the four types, the map  $\eta$  is onto.

To prove that the map  $\eta : \Omega(G'') \rightarrow \Omega(G)$  is 1–1 we have to show that any two rays of  $G''$  that are equivalent in  $G$  are so in  $G''$ . By the definition of ‘contained’ it follows immediately that each two rays of an end of Type 3 or 4, respectively, have equivalent tails in a component of  $\mathcal{C}$ . Since every component of  $\mathcal{C}$  is contained in  $G''$ , each two rays of such an end are equivalent in  $G''$ .

By Lemma 2 each end  $\omega$  of Type 2 is contained in a one-ended component  $C_\omega \in \mathcal{C}^*$  with finite  $N_G(C_\omega)$ . In the construction of  $G''$  for every end of Type 2 a ray  $T_\omega \subseteq C_\omega$  was selected and was linked with  $B$  by a finite path. Since the tree  $B$  consists of  $S^*$ – $S^*$  paths, but the vertex set  $N_G(C_\omega)$  is finite, it follows that  $V(C_\omega) \cap V(B)$  is finite. Hence the set  $V(C_\omega \cap G'') \setminus V(T_\omega)$  is also finite. Since  $\omega$  is contained in  $C_\omega$ , in  $G''$  every ray of  $\omega$  has a common tail with  $T_\omega$ . Thus in  $G''$  any two rays of  $\omega$  are equivalent.

Now let  $\omega$  be an end of Type 1 and let  $T \in \omega$  be a ray in  $G''$ . Then by definition of Type 1 there exist infinitely many pairwise disjoint  $T$ – $S^*$  paths in  $G$ . We show now that there are also infinitely many of such paths in  $G''$ . If  $T$  has a tail in  $B$ , then by construction of  $B$  there exist infinitely many pairwise disjoint  $T$ – $S^*$  paths in  $B$ . Suppose this were not the case and let  $T := x_1x_2\ldots \subseteq B$  be a ray with only finitely many pairwise disjoint  $T$ – $S^*$  paths in  $B$ . Let  $\mathcal{W}$  be a maximal (by inclusion) set of such paths. Then  $V(\mathcal{W})$  is finite and there is a  $j \in \mathbb{N}$  such that no vertex of  $V(\mathcal{W})$  lies in  $B$  above  $x_j$ . But, by construction of  $B$ , the vertex  $x_j$  lies in a path  $W_k$  that joins a vertex  $t_k \in S^*$  with  $\bigcup_{i < k} W_i$ . Thus  $t_k$  lies in  $B$  above  $x_j$ , in contradiction to the assumption. So there are infinitely many pairwise disjoint  $T$ – $S^*$  paths in  $B$  and since  $B \subseteq G''$  they are also in  $G''$ .

If  $B$  contains no tail of  $T$  either there are infinitely many vertices of  $S^*$  in  $T$  and then there are also infinitely many pairwise disjoint  $T$ – $S^*$  paths in  $G''$ , or a tail of  $T$  is contained in a component  $C \in \mathcal{C}$ . Since there are infinitely many pairwise disjoint  $T$ – $S^*$  paths in  $G$ , but the component  $C$  is linked to the rest of the graph only via the vertices of  $S^*$ , all but finitely many of these  $T$ – $S^*$  paths lie in  $C^G$ . Then by Lemma 17 there are also infinitely many of these paths in  $G''$ . Hence by Lemma 7 in  $G''$  the ray  $T$  is equivalent to a ray of  $B$ . So in  $G''$  any two rays  $T, T' \in \omega$  are equivalent to two rays of  $B$  and those have a common tail by Lemma 18. So the two rays of  $B$  are equivalent in  $B$  and thus also in  $G''$ . Hence the rays  $T$  and  $T'$  are also equivalent in  $G''$ .  $\square$

**Lemma 19.** *Let  $S \subseteq V(B)$  be a finite vertex set that separates in  $B$  two vertices  $x, y \in V(B)$ . Then  $S' := \bigcup_{t \in S} [t]$  separates the two vertices in  $G$ .*

*Proof.* If  $S$  separates the two vertices  $x$  and  $y$  in  $B$ , then  $S$  contains a vertex  $v$  of the unique  $x$ – $y$  path  $W$  in  $B$ . This path consists of  $[x] \setminus [y]$ ,  $[y] \setminus [x]$  and the vertex of  $[x] \cap [y]$  with maximal distance to the root. Thus  $[v] \supseteq [x] \cap [y]$  and by Lemma 5 the vertex set  $S'$  separates the vertices  $x$  and  $y$  in  $G$ .  $\square$

**Corollary 3.** *Let  $S \subseteq V(B)$  be a finite vertex set that separates a vertex  $x \in V(B)$  and a ray  $T \subseteq B$  in  $B$ . Then  $S' := \bigcup_{t \in S} [t]$  separates these in  $G$ .*

*Proof.* Since  $S$  is finite, there exists a component of  $B - S$  that contains a tail of  $T$  but not the vertex  $x$ . By Lemma 19 the vertex set  $S'$  separates in  $G$  every vertex of this tail from  $x$ . Since  $S'$  is finite, all (except for finitely many) of these vertices lie in a component of  $G - S'$  that does not contain the vertex  $x$  and form there a tail of  $T$ .  $\square$



**Corollary 4.** *Let  $S \subseteq V(B)$  be a finite vertex set that separates two rays  $T, T' \subseteq B$  in  $B$ . Then  $S' := \bigcup_{t \in S} [t]$  separates these two rays in  $G$ .*

*Proof.* Since  $S$  is finite, there exist a component  $C$  of  $B - S$  that contains a tail  $\tilde{T}$  of  $T$  and a component  $C' \neq C$  of  $B - S$  that contains a tail  $\tilde{T}'$  of  $T'$ . By Lemma 19 the vertex set  $S'$  separates in  $G$  each two vertices of  $\tilde{T}$  and  $\tilde{T}'$ . Since  $S'$  is finite, there exist different components of  $G - S$  that contain tails of  $\tilde{T}$  and  $\tilde{T}'$ , i.e. the vertex set  $S'$  separates  $T$  and  $T'$  in  $G$ .  $\square$

*Second part of the proof of Theorem 1, the uniform continuity of  $\eta^{-1}$ .* In the following we denote by  $\omega, \omega'$  ends of  $G$  and by  $\alpha := \eta^{-1}(\omega), \alpha' := \eta^{-1}(\omega')$  the associated ends of  $G''$ . (Since  $\eta$  is a bijection, this is a unique association.) We now show that the map  $\eta^{-1} : \Omega(G) \rightarrow \Omega(G'')$  is uniformly continuous. Therefore we have to show that for every finite vertex set  $S \subset V(G'')$  there exists a finite vertex set  $\bar{S} \subset V(G)$  that separates those ends in  $G$ , whose associated ends are separated in  $G'' - S$ .

For the construction of the vertex set  $\bar{S}$  we prove the following: Let  $C$  be a component of  $\mathcal{C}$  with infinite  $N_G(C)$ . Moreover, let  $S' \subset V(G)$  be a finite vertex set and let  $\mathcal{K}_{S'}(C)$  be the set of components of  $C - S'$ . Then there exists a finite set  $A_{S'}(C) \subseteq N_G(C)$  such that for all  $K \in \mathcal{K}_{S'}(C)$  we have  $S_K := N_{G-S'}(K) \setminus N_{G''-S'}(K) \subseteq A_{S'}(C)$ . (Note that by the definition of  $\mathcal{K}_{S'}(C)$  we have  $N_{G''-S'}(K) \subseteq N_G(C)$  for all  $K \in \mathcal{K}_{S'}(C)$ .)

First we show that the set  $N_{G-S'}(K)$  is infinite only for finitely many components  $K \in \mathcal{K}_{S'}(C)$ . In order to show this, we prove that every component  $K \in \mathcal{K}_{S'}(C)$  with infinite  $N_{G-S'}(K)$  contains a ray of an end of Type 1. Then this implies together with Lemma 14 the assertion: If infinitely many components of  $\mathcal{K}_{S'}(C)$  contain a ray of an end of Type 1, then there exists a vertex in  $S'$  that lies in the neighbourhood of infinitely many of these components, since  $S'$  is finite. But then  $C$  contains a big star, in contradiction to Lemma 14. Now let  $K \in \mathcal{K}_{S'}(C)$  be an arbitrary component with infinite  $N_{G-S'}(K)$ . We denote by  $K^{G-S'}$  the subgraph of  $G$  that consists of  $K \cup N_{G-S'}(K)$  and all  $K-N_{G-S'}(K)$  edges. Since  $K$  is connected, by Lemma 12 and Lemma 13 this component contains a ray  $T$  with infinitely many pairwise disjoint  $T-S^*$  paths in  $K^{G-S'}$ . By definition this ray belongs to an end of Type 1. Hence  $N_{G-S'}(K)$  is infinite for only finitely many components  $K \in \mathcal{K}_{S'}(C)$ . We now show that the set  $S_K$  is finite for every component  $K \in \mathcal{K}_{S'}(C)$  with infinite  $N_{G-S'}(K)$ . Let  $K \in \mathcal{K}_{S'}(C)$  be such that  $S_K$  is infinite. Since  $K^{G-S'}$  is connected and contains by Lemma 13 no  $v-S^*$  fan, by Lemma 12 there exists a ray  $T$  of  $K$  with infinitely many pairwise disjoint  $T-S_K$  paths in  $K^{G-S'}$ . Clearly, all these paths are contained in  $C^G$ , since  $C^G \supseteq K^{G-S'}$ . Hence, by Lemma 17 there are also infinitely many pairwise disjoint  $T-S_K$  paths in  $C^{G''}$ . Since  $S'$  is finite, there are also infinitely many such paths in  $C^{G''-S'}$ . Then also infinitely many pairwise disjoint  $T-S_K$  paths are contained in  $K^{G''-S'}$ , since the ray  $T$  is contained in  $K$ . But this is a contradiction since in  $G''$  the finite vertex set  $S'$  separates the vertex set  $S_K$  from  $K$ . (Here  $K^{G''-S'}$  and  $C^{G''-S'}$  are defined in the same way as  $C^{G''}$ .) Thus we have shown that the set  $S_K$  is finite for every component  $K \in \mathcal{K}_{S'}(C)$  with infinite  $N_{G-S'}(K)$ . As we have seen above, the set  $N_{G-S'}(K)$  is infinite only for finitely many components  $K \in \mathcal{K}_{S'}(C)$ . Hence there exists a finite set  $A_{S'}^1(C) \subset S^*$  such that for these components we have  $S_K \subseteq A_{S'}^1(C)$ .

Let  $\mathcal{K}'_{S'}(C) := \{K \in \mathcal{K}_{S'}(C) : |N_{G-S'}(K)| < \aleph_0\}$ . Next we show that a finite set  $A^2_{S'}(C) \subset S^*$  exists, such that  $N_{G-S'}(K) \subseteq A^2_{S'}(C)$  for all  $K \in \mathcal{K}'_{S'}(C)$ . Suppose that every subset of  $S^*$  for which this holds is infinite. Then  $\mathcal{K}'_{S'}(C)$  is also infinite, because for each  $K \in \mathcal{K}'_{S'}(C)$  the set  $N_{G-S'}(K)$  is finite. Since the components of  $\mathcal{K}_{S'}(C)$  are disjoint and  $S'$  is finite and  $C$  is connected, there exists a vertex  $v$  of  $C$  that is adjacent to infinitely many components of  $\mathcal{K}'_{S'}(C)$  such that all these components together have infinitely many neighbours in  $S^*$ . Then we can inductively construct a  $v$ - $S^*$  fan in  $C^G$ , since  $N_{G-S'}(K)$  is finite for all  $K \in \mathcal{K}'_{S'}(C)$ . But this is a contradiction to Lemma 13. We now select a finite vertex set  $A^2_{S'}(C) \subset S^*$  such that  $N_{G-S'}(K) \subseteq A^2_{S'}(C)$  for all  $K \in \mathcal{K}'_{S'}(C)$ . Furthermore we set  $A_{S'}(C) := A^1_{S'}(C) \cup A^2_{S'}(C)$ .

Now let  $S$  be an arbitrary finite subset of  $V(G'')$ . For the definition of the vertex set  $\bar{S}$  we define three further vertex sets:

$S^1 := \bigcup_{t \in S} [t]$ ,  $S^2 := \{t_1, \dots, t_l\}$ , where  $l := \max\{r, s, m\}$  with  $t_r := \max_{i \in \mathbb{N}}\{t_i \in S^* \mid \text{there exists } C \in \mathcal{C}^* \text{ with } t_i \in N_G(C) \text{ and } V(C) \cap (S \cup S^1) \neq \emptyset\}$ ,  $t_s := \max_{i \in \mathbb{N}}\{t_i \in S^* \mid \text{there exists } C \in \mathcal{C} \text{ with } t_i \in N_G(C), |N_G(C)| < \aleph_0 \text{ and } V(C) \cap (S \cup S^1) \neq \emptyset\}$  and  $t_m := \max\{t_i \in S^* \cap (S \cup S^1)\}$ . Furthermore let  $\bar{\mathcal{C}} := \{C \in \mathcal{C} \mid |N_G(C)| = \aleph_0, V(C) \cap S \neq \emptyset\}$ . For every  $C \in \bar{\mathcal{C}}$  let the sets  $\mathcal{K}_s(C)$  be defined as above. Since  $S$  is finite and the components of  $\mathcal{C}$  are disjoint,  $\bar{\mathcal{C}}$  is finite. As we have seen before, for each  $C \in \bar{\mathcal{C}}$  there exists a finite set  $A_S(C) \subseteq N_G(C)$  such that  $N_{G-S}(K) \setminus N_{G''-S}(K) \subseteq A_S(C)$  for all  $K \in \mathcal{K}_s(C)$ . Then also  $S^3 := \bigcup_{C \in \bar{\mathcal{C}}} A_S(C)$  is finite. We set  $\bar{S} := S \cup S^1 \cup S^2 \cup S^3$ . Clearly,  $\bar{S}$  is finite.

We now consider two ends  $\alpha$  and  $\alpha'$  of  $G''$  that are separated by  $S$  in  $G''$ . Let  $\omega$  and  $\omega'$  be the corresponding ends of  $G$ . Before we show that the set  $\bar{S}$  separates  $\omega$  and  $\omega'$  in  $G$ , we examine again the different types of ends: If  $\omega$  is an end of Type 1, then  $\omega$  is represented by a ray in  $B$  (see Lemma 7). From now on we denote by  $R_\omega$  such a representing ray of  $\omega$ . To show that  $\bar{S}$  separates in  $G$  the end  $\omega$  and the end  $\omega'$ , it suffices to show that  $\bar{S}$  separates in  $G$  the ray  $R_\omega$  and the end  $\omega'$ .

If  $\omega$  is an end of Type 2, we denote by  $C_\omega$  the component of  $\mathcal{C}^*$  that contains  $\omega$ . By  $T_\omega$  we denote the ray that was selected as representing ray of  $\omega$  in the construction of  $G'$ ; by  $t_{n(\omega)}$  we denote the vertex of  $N_G(C_\omega)$  with maximal index. Note that in the construction of  $G'$  the ray  $T_\omega$  was linked with the vertex  $t_{n(\omega)}$  by a path in  $C_\omega^G$ . (Here  $C_\omega^G$  denotes the subgraph of  $G$  that consists of  $C_\omega \cup N_G(C_\omega)$  and all  $C_\omega$ - $N_G(C_\omega)$  edges of  $G$ .) Hence  $S \cup S^1$  separates the ray  $T_\omega$  and the vertex  $t_{n(\omega)}$  if and only if  $t_{n(\omega)} \in S \cup S^1$  or  $V(C_\omega) \cap (S \cup S^1) \neq \emptyset$ . But then  $N_G(C_\omega) \subseteq \{t_1, \dots, t_{n(\omega)}\} \subseteq S^2 \subseteq \bar{S}$ . Thus, in this case  $\bar{S}$  separates in  $G$  the complete component  $C_\omega$  from the rest of the graph. Since  $C_\omega$  is one-ended, then  $\bar{S}$  separates in  $G$  the end  $\omega$  from all other ends. So in  $G - \bar{S}$  the end  $\omega$  is separated from all other ends whose associated ends are separated in  $G'' - S$  from the end  $\alpha$ . Hence, in the following we consider only the case  $t_{n(\omega)} \notin S \cup S^1$  and  $V(C_\omega) \cap (S \cup S^1) = \emptyset$ . In this case in  $G - (S \cup S^1)$  the end  $\omega$  is contained in the same component as the vertex  $t_{n(\omega)}$ . The same holds for the corresponding end  $\alpha$  in  $G'' - S$ . This means if  $S$  separates in  $G''$  the end  $\alpha$  and the end  $\alpha'$ , then  $S$  separates the vertex  $t_{n(\omega)}$  and the end  $\alpha'$ . To show that  $\bar{S}$  separates in  $G$  the end  $\omega$  and the end  $\omega'$  it suffices to show that  $S \cup S^1$  separates in  $G$  the vertex  $t_{n(\omega)}$  and the end  $\omega'$ .

If  $\omega$  is an end of Type 3, then we denote by  $C_\omega$  the component of  $\mathcal{C}$  that contains  $\omega$  and by  $K_\omega$  the component of  $C_\omega - S$  that contains  $\omega$ . Furthermore let  $t_{n(\omega)}$  be the vertex of  $N_G(C_\omega)$  with maximal index (by the definition of Type 3 the set

$N_G(C_\omega)$  is finite). If  $\omega$  is separated in  $G'' - (S \cup S^1)$  from the vertex  $t_{n(\omega)}$ , either  $t_{n(\omega)} \in S \cup S^1$  or  $V(C) \cap (S \cup S^1) \neq \emptyset$ . In both cases it follows from the definition of  $S^2$  that  $N_G(C_\omega) \subseteq \{t_1, \dots, t_{n(\omega)}\} \subseteq S^2 \subseteq \bar{S}$ , which means that in  $G$  the vertex set  $\bar{S}$  separates the complete component  $C$  from the rest of the graph. Since  $\bar{S} \supseteq S$ ,  $C - \bar{S}$  decomposes into finer components than  $C - S$ . So in  $G - \bar{S}$  the end  $\omega$  is separated from all other ends whose associated ends are separated in  $G'' - S$  from the end  $\alpha$ . Thus in the following we consider only the case  $V(C) \cap (S \cup S^1) = \emptyset$  and  $t_{n(\omega)} \notin S \cup S^1$ , which means that in  $G'' - S$  and in  $G - (S \cup S^1)$  the vertex  $t_{n(\omega)}$  is adjacent to the whole component  $C$ . So, if  $S$  separates in  $G''$  the end  $\alpha$  and the end  $\alpha'$ , then  $S$  separates the vertex  $t_{n(\omega)}$  and  $\alpha'$ . To show that  $\bar{S}$  separates in  $G$  the end  $\omega$  and the end  $\omega'$  it suffices to show that  $S \cup S^1$  separates in  $G$  the vertex  $t_{n(\omega)}$  and the end  $\omega'$ .

If  $\omega$  is an end of Type 4, then let  $C_\omega$  be the component of  $\mathcal{C}$  that contains  $\omega$ . Let  $K_\omega$  be the component of  $C_\omega - S$  that, considered as subgraph of  $G$ , contains  $\omega$ . Let  $\bar{K}_\omega$  be the corresponding component of  $C_\omega - \bar{S}$ . Since  $S \subseteq \bar{S}$ , it follows that  $K_\omega \supseteq \bar{K}_\omega$ .

We now show that  $N_{G''-S}(K_\omega) \supseteq N_{G-\bar{S}}(\bar{K}_\omega)$ . By the construction of  $G''$  we have  $N_{G''}(C) = N_G(C)$  for each component  $C \in \mathcal{C}$  with infinite  $N_G(C)$ . Hence  $V(C) \cap S \neq \emptyset$  if  $N_{G-S}(C) \setminus N_{G''-S}(C) \neq \emptyset$ . As shown above, we have  $A_S(C) \supseteq N_{G-S}(C) \setminus N_{G''-S}(C)$  for every component  $C$  of  $C - S$ , thus also for the component  $K_\omega$ . But this means  $N_{G''-S}(K_\omega) \supseteq N_{G-S}(K_\omega) \setminus A_S(C)$ . Since  $A_S(C) \subseteq S^3 \subseteq \bar{S}$ , it follows that  $N_{G''-S}(K_\omega) \supseteq N_{G-\bar{S}}(K_\omega)$ . So the component  $\bar{K}_\omega$  is in  $G - \bar{S}$  linked with  $G \setminus \bar{K}_\omega$  only by vertices of  $N_{G''-S}(K_\omega)$ ; the same holds for the component  $K_\omega$  in  $G'' - S$ . If  $S$  separates in  $G''$  the end  $\alpha'$  and the end  $\alpha$ , then  $\alpha'$  is contained in a component of  $G'' - S$  that is different from  $K_\omega$  and  $S$  separates the vertex set  $N_{G''-S}(K_\omega)$  and the end  $\alpha'$ . Since  $\bar{K}_\omega \subseteq K_\omega$ , also  $\omega'$  is not contained in  $\bar{K}_\omega$  but in another component of  $G - \bar{S}$ . Thus, for the proof that  $\bar{S}$  separates in  $G$  the end  $\omega'$  and the end  $\omega$  it suffices to show that  $\bar{S}$  separates in  $G$  the vertex set  $N_{G''-S}(K_\omega)$  and the end  $\omega'$ .

We show now that  $\bar{S}$  separates  $\omega$  and  $\omega'$  in  $G$ . By the above considerations it suffices, depending on the type of ends, to show that  $\bar{S}$  separates in  $G$  two vertex sets of the type  $N_{G''-S}(K_\omega)$  or  $\{t_{n(\omega)}\}$ , or separates such a vertex set and a ray, or separates two rays. But, since  $\bar{S} \supseteq S \cup S^1$ , this is proved by Lemma 19 and Corollaries 3 and 4. So the proof of Theorem 1 is complete.  $\square$

## REFERENCES

1. J.-M. Brochet and R. Diestel, *Normal Tree Orders for Infinite Graphs*. Transactions of the American Mathematical Society, Vol. 345, No. 2, pp. 871–895, 1994. MR **95a**:05024
2. R. Diestel. *The End Structure of a Graph: Recent Results and Open Problems*. Discrete Mathematics Vol. 100, pp. 313–327, 1992. MR **93g**:05038
3. R. Diestel, *On Spanning Trees and  $k$ -connectedness in Infinite Graphs*. Journal of Combinatorial Theory, Series B 56, 1992. MR **93i**:05047
4. R. Halin, *Über unendliche Wege in Graphen*. Mathematische Annalen 157, 1964. MR **30**:578
5. R. Halin, *Simplicial decompositions in infinite graphs*. Advances in Graph Theory (B. Bollobás, Ed.), Annals of Discrete Mathematics 3, North-Holland Publ. Co., Amsterdam / London 1978. MR **80a**:05162
6. R. Halin, *Graphentheorie I, II*. Wissenschaftliche Buchgesellschaft, Darmstadt 1981. MR **81m**:05001; MR **84h**:05001
7. I.M. James, *Topological and Uniform Spaces*. Springer Verlag, New York, 1987. MR **89b**:54001

8. H.A. Jung, *Wurzelbäume und unendliche Wege in Graphen*. Mathematische Nachrichten 41, 1969. MR **42**:1710
9. D. König, *Theorie der endlichen und unendlichen Graphen*. Akademische Verlagsgesellschaft, Leipzig 1936 (reprinted: Chelsea, New York, 1950). MR **12**:195g
10. H. Schubert, *Topologie*. B.G. Teubner Verlag, Stuttgart, 1964. MR **30**:551
11. P.D. Seymour and R. Thomas, *An end-faithful counterexample*. Directions in Infinite Graph Theory and Combinatorics (R. Diestel, Ed.), Topics in Discrete Mathematics 3, North-Holland Publ. Co., Amsterdam / London, 1992. CMP 92:14
12. C. Thomassen, *Infinitely connected graphs with no end-preserving spanning trees*. preprint, 1990.
13. K. Waas, *Topologisch endentreue Bäume in unendlichen Graphen*. Diplomarbeit, Universität Bielefeld, 1994.

FAKULTÄT FÜR MATHEMATIK, TU CHEMNITZ D-09107 CHEMNITZ, GERMANY